MATH 140A Review: Mathematical Induction

1. Show that $6 \mid (11^n - 5^n)$ for $n = 1, 2, \ldots$.

Solution:

Proof. We need to show that

$$6 \mid (11^n - 5^n) \text{ for } n = 1, 2, 3, \ldots. \tag{1}$$

(basis step) Since $11^1 - 5^1 = 6$, (1) holds for $n = 1$.

(induction step) Assume (1) holds for $n = k$. That is, $6 \mid (11^k - 5^k)$. Our goal is to show (1) holds for $n = k + 1$, that is $6 \mid (11^{k+1} - 5^{k+1})$. Since $6 \mid (11^k - 5^k)$, then $11^k - 5^k = 6q$ for some $q \in \mathbb{Z}$. We may rewrite the previous to get $11^k = 6q + 5^k$. Thus,

$$11^{k+1} - 5^{k+1} = 11 \cdot (6q + 5^n) - 5 \cdot 5^k = 5^k(11 - 5) + 11 \cdot 6q = 6 \cdot (5^k + 11q).$$

Thus, (1) holds for $n = k + 1$. By mathematical induction,

$$6 \mid (11^n - 5^n) \text{ for } n = 1, 2, 3, \ldots$$

□

2. Consider the sequence defined by

$$a_1 = 2 \quad \text{and} \quad a_{n+1} = 2^{a_n} \text{ for } n \geq 1.$$

Show that $a_n \leq a_{n+1}$ for $n = 1, 2, 3 \ldots$.

Solution:

Proof. We need to show that

$$a_n \leq a_{n+1} \text{ for } n = 1, 2, 3, \ldots \tag{2}$$

(basis step) Since $a_1 = 2$ and $a_2 = 4$, (1) holds for $n = 1$.

(induction step) Assume (1) holds for $n = k$. That is, $a_k \leq a_{k+1}$. Since the function 2^x is increasing, then $a_k = 2^{a_k} \leq 2^{a_{k+1}} = a^{k+1}$. Thus, (2) holds for $n = k + 1$. By mathematical induction, (2) holds for $n = 1, 2, \ldots$.

□

3. Consider the sequence defined by

$$a_1 = \sqrt{3} \quad \text{and} \quad a_{n+1} = \sqrt{a_n} \text{ for } n \geq 1.$$

Show that $a_n \geq a_{n+1}$ for $n = 1, 2, 3 \ldots$.

Solution:

Proof. We need to show that

$$a_n \geq a_{n+1} \text{ for } n = 1, 2, 3, \ldots. \tag{3}$$

(basis step) Since $a_1 = \sqrt{3}$ and $a_2 = 3^{1/4}$, then (1) holds for $n = 1$.

(induction step) Assume (3) holds for $n = k$. That is, $a_k \geq a_{k+1}$. Since the function $\sqrt{x}$ is increasing, $\sqrt{a_k} \geq \sqrt{a_{k+1}}$. That is, $a_{k+1} \geq a_{k+2}$. Thus, (3) holds for $n = k + 1$. By mathematical induction, (3) holds for $n = 1, 2, \ldots$.

□